Models of Set Theory I – Summer 2017

Prof. Peter Koepke, Dr. Philipp Lücke – Problem Sheet 10

Problem 37 [4 points]

- A *σ-algebra* is a collection of subsets of the reals which is closed under countable unions and complements (and hence also under countable intersections).
- The *Borel sets* are the elements of the smallest σ-algebra containing all basic open sets of the form I_s for $s \in {}^{<\omega}2$.
- A set of reals X has the *property of Baire* if there is an open set O such that the symmetric difference $X \triangle O$ is meager.

Show that every Borel set has the property of Baire.

Problem 38 [6 points] Let B denote the collection of Borel sets. With the operations of intersection, union, and complement, B forms a complete Boolean algebra. We define an equivalence relation $\sim$ on B by setting, for $X, Y \in B$, $X \sim Y$ in case $X \triangle Y$ is meager. Let D be the collection of equivalence classes $[X]_\sim$ of B with respect to $\sim$. Verify the following.

- D is a complete Boolean algebra, with the operations induced from B.
- For every nonmeager Borel set B, there is $s \in {}^{<\omega}2$ such that $[I_s]_\sim \leq [B]_\sim$.

Hint: Use the result of Problem 37.

- D is (modulo isomorphism) a completion of Cohen forcing.

Problem 39 [6 points] Let $N \in \omega$, and let $\vec{n} = \langle n_i \mid i < N \rangle$ and $\vec{k} = \langle k_i \mid i < N \rangle$ be increasing sequences of natural numbers, where we require $\vec{n}$ to be strictly increasing, and we require that $\forall i < N \; n_i < k_i$. Show that if ZFC + GCH is consistent, then so is ZFC plus

$$\forall i < N \; 2^{\aleph_{n_i}} = \aleph_{k_i}.$$

Hint: Perform N successive forcing constructions, thereby handling the n_i in reverse order.

Problem 40 [4 points] Work over a countable ground model M. Let P be the forcing $\mathrm{Fn}(\aleph_1, \mathcal{P}(\omega), \aleph_0)$, as defined in M.

- Which M-cardinals are preserved by forcing with P?
- What is the value of $2^{\aleph_0}$ in P-generic extensions of M?